Properties of some character tables related to the symmetric groups

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Abstract

We determine invariants like the Smith normal form and the determinant for certain integral matrices which arise from the character tables of the symmetric groups S_n and their double covers. In particular, we give a simple computation, based on the theory of Hall-Littlewood symmetric functions, of the determinant of the regular character table $\mathcal{X}_{RC}$ of S_n with respect to an integer $r \geq 2$. This result had earlier been proved by Olsson in a longer and more indirect manner. As a consequence, we obtain a new proof of the Mathas' Conjecture on the determinant of the Cartan matrix of the Iwahori-Hecke algebra. When r is prime we determine the Smith normal form of $\mathcal{X}_{RC}$. Taking r large yields the Smith normal form of the full character table of S_n. Analogous results are then given for spin characters.

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[1]Partially supported by The Danish National Research Council.
[2]Partially supported by NSF grant #DMS-9988459.

1 Introduction

In this paper we determine invariants like the Smith normal form or the determinant for certain integral matrices which come from the character tables of the finite symmetric groups S_n and their double covers $\hat{S}_n$. The matrices in question are the so-called regular and singular character tables of S_n and the reduced spin character table of $\hat{S}_n$.

In section 2 we calculate the determinants of the r-regular and r-singular character tables of S_n for arbitrary integers $r \geq 2$, using symmetric functions and some bijections involving regular partitions. The knowledge of these determinants is equivalent to the knowledge of the determinants of certain "generalized Cartan matrices" of S_n as considered in [9]. In particular we obtain a new proof of a conjecture of Mathas about the Cartan matrix of an Iwahori-Hecke algebra of S_n at a primitive rth root of unity which is simpler than the original proof given by Brundan and Kleshchev in [3]. In section 3 we determine the Smith normal form of the regular character table in the case where r is a prime. As a special case the Smith normal form of the character table of S_n may be calculated. We also determine the Smith normal form of the reduced spin character table for $\hat{S}_n$. The paper also presents some open questions.

2 The determinant of the regular part of the character table of S_n

We fix positive integers n, r, where $r \geq 2$.

If $\mu = (\mu_1, \mu_2, \ldots)$ is a partition of n we write $\mu \in \mathcal{P}$ and denote by $\ell(\mu)$ the number of (non-zero) parts of μ. We let z_μ denote the order of the centralizer of an element of (conjugacy) type μ in S_n. Suppose $\mu = (1^{m_1(\mu)}, 2^{m_2(\mu)}, \ldots)$, is written in exponential notation. Then we may factor $z_\mu = a_\mu b_\mu$, where

$$a_\mu = \prod_{i \geq 1} i^{m_i(\mu)}, \quad b_\mu = \prod_{i \geq 1} m_i(\mu)!$$

Whenever $\mathcal{Q} \subseteq \mathcal{P}$ we define

$$a_{\mathcal{Q}} = \prod_{\mu \in \mathcal{Q}} a_\mu, \quad b_{\mathcal{Q}} = \prod_{\mu \in \mathcal{Q}} b_\mu.$$

Let $\mu \in \mathcal{P}$. We write $\mu \in R$ and call μ *regular* if $m_i(\mu) \leq r - 1$ for all $i \geq 1$. We write $\mu \in C$ and call μ *class regular* if $m_i(\mu) = 0$, whenever $r \mid i$.

We are particularly interested in the integers a_C and b_C. By [12, Theorem 4] there is a connection between a_C and b_C given by

$$b_C = r^{d_C} \, a_C, \tag{1}$$

where the class regular defect number d_C is defined by

$$d_C = \sum_{\mu \in C} d(\mu), \quad d(\mu) = \sum_{i,k \geq 1} \left\lfloor \frac{m_i(\mu)}{r^k} \right\rfloor .$$

Here $\lfloor \cdot \rfloor$ is the floor function, i.e., $\lfloor x \rfloor$ denotes the integral part of x. Note that for $r > n$ we have $R = C = \mathcal{P}$ and then $d_{\mathcal{P}} = 0$ and thus $a_{\mathcal{P}} = b_{\mathcal{P}}$.

Let $\mathcal{X}_{RC}$ denote the regular character table of S_n with respect to r. It is a submatrix of the character table $\mathcal{X}$ of S_n. The subscript RC indicates that the rows of $\mathcal{X}_{RC}$ are indexed by the set R of regular partitions of n, and the columns by the set C of class regular partitions of n. We want to present a proof of the following result:

Theorem 1 *We have*

$$|\det(\mathcal{X}_{RC})| = a_C.$$

This result was first proved in [12], but the proof relied on results of [9] for which the work of Donkin [4] and Brundan and Kleshchev [3] was used in a crucial way. Our proof of Theorem 1 does not use [4] or [3]; it is direct and thus much shorter.

In [9], an r-analogue of the modular representation theory for S_n was developed systematically, and in particular, an r-analogue of the Cartan matrix for the symmetric groups (and the corresponding r-blocks) was introduced.

In [2] the explicit value of this latter determinant was conjectured to be r^{d_C} in the notation above; this was proved in [9, Proposition 6.11] using [4] and [3]. This result is now a consequence of our theorem:

Corollary 2 *Let $\mathcal{C}$ be the r-analogue of the Cartan matrix of S_n as defined in [9]. Then we have*

$$\det(\mathcal{C}) = r^{d_C} .$$

Proof. As is shown in [12] there is a simple equation connecting the determinants of $\mathcal{C}$ and $\mathcal{X}_{RC}$, namely

$$\det(\mathcal{X}_{RC})^2 \det(\mathcal{C}) = a_C b_C .$$

Thus in view of equation (1) Theorem 1 implies the Corollary. $\square$

Mathas conjectured that the determinant of the Cartan matrix of an Iwahori-Hecke algebra of S_n at a primitive rth root of unity should be a power of r; via [4], the conjecture in [2] mentioned above predicted the explicit value of this determinant, thus providing a strengthening of Mathas' conjecture. Mathas' conjecture was proved by Brundan and Kleshchev [3]; in fact, they also gave an explicit formula for this determinant for blocks of the Hecke algebra. We can now provide an alternative proof of these conjectures.

Corollary 3 *The strengthened Mathas' conjecture is true.*

Proof. Donkin [4] has shown that the Cartan matrix for the Hecke algebra has the same determinant as the Cartan matrix $\mathcal{C}$ considered in Corollary 2. $\square$

Based on this and the results on r-blocks in [9], the results in [2] then also give the determinants of Cartan matrices of r-blocks of S_n explicitly, without the use of [3].

Let us finally mention that in [9, Section 6] there is an explicit conjecture about the Smith normal form of $\mathcal{C}$. In the case where r is a prime, this is known to be true by the general theory of R. Brauer. One may also ask about the Smith normal form of $\mathcal{X}_{RC}$; we answer this question in this article in the prime case.

We now proceed to describe the proof of Theorem 1. It is obtained by combining Theorems 4 and 5 below. Theorem 4 evaluates $\det (\mathcal{X}_{RC})^2$ using symmetric functions as an expression involving a primitive rth root of unity. Theorem 5 shows that this expression equals ${a_C}^2$. It is based on general bijections involving regular partitions.

Define

$$z_\mu(t) = z_\mu \prod_j (1-t^{\mu_j})^{-1} = z_\mu \prod_i (1-t^i)^{-m_i(\mu)}$$

where the product ranges over all j for which $\mu_j > 0$, and

$$b_\lambda(t) = \prod_i (1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)}).$$

Let $\omega = e^{2\pi i/r}$, a primitive rth root of unity.

We use notation from the theory of symmetric functions from [10] or [14]. In particular, m_λ, s_λ, and p_λ denote the monomial, Schur, and power sum symmetric functions, respectively, indexed by the partition λ.

Theorem 4 *We have*

$$\det(\mathcal{X}_{RC})^2 = \prod_{\mu\in C} z_\mu(\omega)\cdot\prod_{\lambda\in R} b_\lambda(\omega).$$

Proof. Let $Q_\lambda(x;t)$ denote a Hall-Littlewood symmetric function as in [10, p. 210]. It is immediate from the definition of $Q_\lambda(x;t)$ that $Q_\lambda(x;\omega)=0$ unless $\lambda\in R$. Moreover (see [10, Exam. III.7.7, p. 249]) when $Q_\lambda(x;\omega)$ is expanded in terms of power sums p_μ, only class regular μ appear. Thus [10, (7.5), p. 247] for $\lambda\in R$ we have

$$Q_\lambda(x;\omega) = \sum_{\mu\in C} z_\mu(\omega)^{-1} X^\lambda_\mu(\omega) p_\mu(x),$$

where $X^\lambda_\mu(t)$ is a Green's polynomial.

Hence by [10, (7.4)] the matrix $X(\omega)_{RC} = (X^\lambda_\mu(\omega))$, where $\lambda\in R$ and $\mu\in C$, satisfies

$$\det(X(\omega)_{RC})^2 = \prod_{\mu\in C} z_\mu(\omega)\prod_{\lambda\in R} b_\lambda(\omega). \tag{2}$$

Now consider the symmetric function $S_\lambda(x;t)$ as defined in [10, (4.5), p. 224]. It follows from the formula $S_\lambda(x;t)=s_\lambda(\xi)$ in [10, top of p. 225] that

$$S_\lambda(x;t) = s_\lambda(p_j \to (1-t^j)p_j),$$

i.e., expand $s_\lambda(x)$ as a polynomial in the p_j's and substitute $(1-t^j)p_j$ for p_j. Since

$$s_\lambda = \sum_\mu z_\mu^{-1}\chi^\lambda(\mu)p_\mu,$$

we have

$$S_\lambda(x;\omega) = \sum_{\mu\in C} z_\mu(\omega)^{-1}\chi^\lambda(\mu)p_\mu.$$

The $S_\lambda(x;\omega)$'s thus lie in the space $A_{(r)}$ spanned over $\mathbb{Q}(\omega)$ by the p_μ's where $\mu\in C$. Since the $Q_\mu(x;\omega)$'s for regular μ span $A_{(r)}$ by [10, Exam. III.7.7, p. 249], the same is true of the $S_\lambda(s;\omega)$'s. Moreover, the transition matrix $M(S,Q)_{RR}$ between the $Q_\lambda(x;t)$'s and $S_\lambda(x;t)$'s is lower unitriangular by [10, top of p. 239] and [10, p. 241]. Hence

$$\det M(S,Q)_{RR} = 1. \tag{3}$$

Let $M(S,p)_{RC}$ denote the transition matrix from the p_μ's to S_λ's for $\mu \in C$ and $\lambda \in R$. Let $Z(t)_{CC}$ denote the diagonal matrix with entries $z_\lambda(t)$, $\lambda \in C$. By the discussion above we have

$$\begin{aligned}
\mathcal{X}_{RC} &= M(S,p)_{RC}\, Z(\omega)_{CC} \quad \text{(by the relevant definitions)} \\
&= M(S,Q)_{RR}\, M(Q,p)_{RC}\, Z(\omega)_{CC} \\
&= M(S,Q)_{RR}\, X(\omega)_{RC}\, Z(\omega)_{CC}^{-1}\, Z(\omega)_{CC} \\
&= M(S,Q)_{RR}\, X(\omega)_{RC}.
\end{aligned}$$

Taking determinants and using (2) and (3) completes the proof. $\square$

Define

$$A_C(\omega) = \prod_{\mu \in C} \prod_i (1-\omega^i)^{-m_i(\mu)}$$

$$B_R(\omega) = \prod_{\lambda \in R} b_\lambda(\omega)^{-1} = \prod_{\lambda \in R} \left(\prod_i (1-\omega)(1-\omega^2)\cdots(1-\omega^{m_i(\lambda)}) \right)^{-1},$$

so that by Theorem 4

$$\det(\mathcal{X}_{RC})^2 = a_C b_C A_C(\omega) B_R(\omega)^{-1}.$$

In order to complete the proof of Theorem 1 we thus just need to show:

$$\frac{B_R(\omega)}{A_C(\omega)} = \frac{b_C}{a_C}.$$

As $\dfrac{b_C}{a_C} = r^{d_C}$ this is equivalent to showing

Theorem 5 *We have*

$$\frac{B_R(\omega)}{A_C(\omega)} = r^{d_C}.$$

Clearly the factors $1-\omega^j$ occurring on the left hand side in Theorem 5 depend only on the residue of j modulo r. Thus

$$A_C(\omega)^{-1} = \prod_{s=1}^{r-1} (1-\omega^s)^{\alpha_C^{(s)}}, \quad B_R(\omega)^{-1} = \prod_{s=1}^{r-1} (1-\omega^s)^{\beta_R^{(s)}},$$

where

$$\alpha_C^{(s)} = \sum_{\mu \in C} \sum_{\{i \mid i \equiv s \pmod r\}} m_i(\mu)$$

$$\beta_R^{(s)} = \sum_{\rho\in R} |\{i|m_i(\rho) \geq s\}|.$$

We use the bijections $\kappa^{(s)}$ defined in Proposition 9 below to show the following:

Proposition 6 *For all* $s \in \{1, \ldots, r-1\}$ *we have*

$$\alpha_C^{(s)} = \beta_R^{(s)} + d_C.$$

This shows then that

$$\frac{B_R(\omega)}{A_C(\omega)} = \left(\prod_{s=1}^{r-1}(1-\omega^s)\right)^{d_C}.$$

Then Theorem 5 follows from the fact that

$$\prod_{s=1}^{r-1}(1-\omega^s) = r.$$

(Simply substitute $x = 1$ in the identity $1 + x + \cdots + x^{r-1} = \prod_{s=1}^{r-1}(x - \omega^s)$.)

Let $m \in \mathbb{N}$. We write m in its r-adic decomposition as $m = \sum_{j\geq 0} m_j r^j$, i.e., with $m_j \in \{0, \ldots, r-1\}$ for all j. For $m \neq 0$, we can write $m = \sum_{j\geq k} m_j r^j$, with $m_k \neq 0$. In the power series convention, $k(m) = k$ is the degree of m and $\ell(m) = m_k$ its leading coefficient. We also set $h(m) = \sum_{j\geq k+1} m_j r^j = r^{k+1} q(m)$ for the higher terms of m. Thus

$$m = \ell(m) r^{k(m)} + q(m) r^{k(m)+1} .$$

For a given a, we define

$$h_a(m) = \sum_{j\geq a} m_j r^j = q_a(m) r^a \ , \quad q_a(m) = \left\lfloor \frac{m}{r^a} \right\rfloor .$$

We call $e \in \{1, \ldots, m\}$ a *non-defect number for* m, if $h(e) = h_{k(e)+1}(m)$, otherwise e is a *defect number for* m (and then $h(e) < h_{k(e)+1}(m)$, and hence $q(e) < q_{k(e)+1}(m)$). Thus the non-defect numbers for m are of the form

$$e = e_a r^a + h_{a+1}(m) \ , \quad e_a \in \{1, \ldots, m_a\} ,$$

and thus there are $\sum_{j\geq 0} m_j$ such numbers. The defect numbers for m are of the form

$$e = e_a r^a + q r^{a+1} \ , \quad e_a \in \{1, \ldots, r-1\} \ , \ q \in \{0, \ldots, q_{a+1}(m) - 1\} .$$

Their parameters (a, q) thus belong to the set

$$\mathcal{D}(m) = \{(a, q) \mid a \geq 0,\ 0 \leq q < q_{a+1}(m)\},$$

which is of cardinality

$$d(m) = \sum_{a \geq 1} \left\lfloor \frac{m}{r^a} \right\rfloor ,$$

called the *defect* of m. For each $s \in \{1, \ldots, r-1\}$ there are exactly $d(m)$ defect numbers for m with leading coefficient s, namely $e = sr^a + qr^{a+1}$, where $(a, q) \in \mathcal{D}(m)$. Thus clearly we have $(r-1)d(m)$ defect numbers for m and

$$m = (r-1)d(m) + \sum_{j \geq 0} m_j .$$

For $\mu \in \mathcal{P}$, its defect (as defined at the beginning of this section) is then

$$d(\mu) = \sum_{i \geq 1} d(m_i(\mu)).$$

For $s \in \{1, \ldots, r-1\}$ set

$$\mathcal{D}^{(s)}(\mu) = \{(i, a, q) \mid \ell(i) = s, (a, q) \in \mathcal{D}(m_i(\mu))\}$$

and

$$\mathcal{D}(\mu) = \bigcup_{s=1}^{r-1} \mathcal{D}^{(s)}(\mu) .$$

We have that

$$d^{(s)}(\mu) = |\mathcal{D}^{(s)}(\mu)| = \sum_{\{i \geq 1, \ell(i) = s\}} d(m_i(\mu))$$

and

$$d(\mu) = \sum_{s=1}^{r-1} d^{(s)}(\mu) = |\mathcal{D}(\mu)| .$$

Consider nonzero residues s, t modulo r, let $\mu = (i^{m_i(\mu)})$ and define

$$\mathcal{T}^{(st)}(\mu) = \{(i, j) \mid 1 \leq i,\ 1 \leq j \leq m_i(\mu), \ell(i) = s, \ell(j) = t\}.$$

Glaisher [6] defined a bijection between the sets C and R of class regular and regular partitions of n. Glaisher's map G is defined as follows. Suppose

that $\mu = (i^{m_i(\mu)}) \in C$. Consider the r-adic expansion of each multiplicity $m_i(\mu)$:

$$m_i(\mu) = \sum_{j \geq 0} m_{ij}(\mu) r^j$$

where for all relevant i, j we have $m_{ij}(\mu) \in \{0, \ldots, r-1\}$. Then $G(\mu) = \rho$ where for all $i, j, r \nmid i$ we have $m_{ir^j}(\rho) = m_{ij}(\mu)$.

We show

Proposition 7 *If $\mu \in C$ then $|\mathcal{T}^{(st)}(\mu)| = |\mathcal{T}^{(st)}(G(\mu))| + d^{(s)}(\mu)$.*

Proof. We establish a bijection $\delta^{(st)}(\mu)$ between $\mathcal{T}^{(st)}(\mu)$ and the disjoint union $\mathcal{T}^{(st)}(G(\mu)) \cup \mathcal{D}^{(s)}(\mu)$. If $(i, j) \in \mathcal{T}^{(st)}(\mu)$ and $(k(j), q(j)) = (a, q)$, we have two possibilities

(i) j is a defect number for $m_i(\mu)$. Then we map (i, j) onto $(i, a, q) \in \mathcal{D}^{(s)}(\mu)$.

(ii) We have $j = tr^a + h_{a+1}(m_i(\mu))$ where $1 \leq t \leq m_{ia}(\mu)$. Then we map (i, j) onto $(r^a i, t) \in \mathcal{T}^{(st)}(G(\mu))$.

This establishes the desired bijection. □

Consider nonzero residues s, t modulo r, and define

$$\mathcal{T}_C^{(st)} = \{(\mu, i, j) | \mu \in C, (i, j) \in \mathcal{T}^{(st)}(\mu)\}$$
$$\mathcal{T}_R^{(st)} = \{(\rho, i, j) | \rho \in R, (i, j) \in \mathcal{T}^{(st)}(\rho)\}$$
$$\mathcal{D}^{(s)} = \{(\mu, i, a, q) | \mu \in C, (i, a, q) \in \mathcal{D}^{(s)}(\mu)\}.$$

Clearly the bijections $\delta^{(st)}(\mu),\ \mu \in C$, above induce a bijection

$$\delta^{(st)} : \mathcal{T}_C^{(st)} \longleftrightarrow \mathcal{T}_R^{(st)} \cup \mathcal{D}^{(s)}.$$

Putting the bijections $\delta^{(ts)},\ t = 1, \ldots, r-1$ together we obtain a bijection

$$\delta^{(s)} : \bigcup_{t=1}^{r-1} \mathcal{T}_C^{(ts)} \longleftrightarrow \bigcup_{t=1}^{r-1} \mathcal{T}_R^{(ts)} \cup \mathcal{C},$$

where

$$\mathcal{C} = \bigcup_{t=1}^{r-1} \mathcal{D}^{(t)}.$$

In [12, proof of Theorem 4], an involution ι was defined on the set

$$\mathcal{T}_C = \{(\mu, i, j) | \mu \in C, i, j \geq 1, m_i(\mu) \geq j\}.$$

From the definition of ι it follows that it maps the subset $\mathcal{T}_C^{(st)}$ of $\mathcal{T}_C$ into $\mathcal{T}_C^{(ts)}$. Thus we conclude

Lemma 8 *For all $s \in \{1, \ldots, r-1\}$ there is a bijection*

$$\iota^{(s)} : \bigcup_{t=1}^{r-1} \mathcal{T}_C^{(st)} \longleftrightarrow \bigcup_{t=1}^{r-1} \mathcal{T}_C^{(ts)} .$$

Composing the bijections $\iota^{(s)}$ and $\delta^{(s)}$ we see

Proposition 9 *For all $s \in \{1, \ldots, r-1\}$ there is a bijection*

$$\kappa^{(s)} : \bigcup_{t=1}^{r-1} \mathcal{T}_C^{(st)} \longleftrightarrow \bigcup_{t=1}^{r-1} \mathcal{T}_R^{(ts)} \cup \mathcal{C}.$$

Proof of Proposition 6. Just consider the cardinalities of the sets occurring in Proposition 9.

$$\left| \bigcup_{t=1}^{r-1} \mathcal{T}_C^{(st)} \right| = \sum_{\mu \in C} \sum_{\{i|\ell(i)=s\}} m_i(\mu) = \alpha_C^{(s)}.$$

The latter equality holds because a class regular partition contains no parts divisible by r. Thus if $m_i(\mu) \neq 0$ then $\ell(i) = s$ if and only if $i \equiv s (\mathrm{mod}\ r)$.

$$\left| \bigcup_{t=1}^{r-1} \mathcal{T}_R^{(ts)} \right| = \sum_{\rho \in R} |\{i|m_i(\rho) \geq s\}| = \beta_R^{(s)}.$$

This is because parts in regular partitions have multiplicities $< r$. Finally

$$|\mathcal{C}| = \sum_{t=1}^{r-1} d^{(t)} = \sum_{\mu \in C} d(\mu) = d_C. \quad \square$$

Remark. There is of course also a *singular* character table for S_n, which we denote $\mathcal{X}_{R'C'}$. It is also a submatrix of the character table $\mathcal{X}$ of S_n. The subscript $R'C'$ indicates that the rows of $\mathcal{X}_{R'C'}$ are indexed by the set R' of singular (i.e. nonregular) partitions of n, and the columns by the set C' of class singular (i.e. non-class regular) partitions of n. For this we have

$$|\det(\mathcal{X}_{R'C'})| = b_{C'}. \tag{4}$$

There are different ways of proving this. In [12] there is a proof based on Theorem 1 and a result in [9].

Another way of proving (4) is *via* an identity of Jacobi [5, p. 21]. Namely, suppose that A is an invertible $n \times n$ matrix, and write A and A^{-1} in the block form

$$A = \left[\begin{array}{cc} B & C \\ D & E \end{array}\right], \quad A^{-1} = \left[\begin{array}{cc} B' & C' \\ D' & E' \end{array}\right],$$

where B and B' are $k \times k$ matrices. Then

$$\det E' = \frac{\det B}{\det A}.$$

By the orthogonality of characters we have

$$\mathcal{X}^{-1} = \mathcal{X}^t \Delta(z_\mu^{-1}),$$

where $\Delta(z_\mu^{-1})$ is the diagonal matrix with the $z_\mu^{-1},\ \mu \in \mathcal{P}$, on the diagonal. Equation (4) follows immediately from this observation and Theorem 1.

Remark. If we keep r fixed and let n vary, then the result of Proposition 6 may also be proved by calculating the generating functions for $\alpha_C^{(s)}, \beta_R^{(s)}$ and d_C. Indeed, if $P(q)$ is the generating function for the number of partitions of n, then $P_r(q) = \frac{P(q)}{P(q^r)}$ is the generating function for the number of regular partitions of n. We may then express the generating functions for $\alpha_C^{(s)}, \beta_R^{(s)}$ and d_C respectively by

$$A^{(s)}(q) = P_r(q) \sum_{i \geq 0} \frac{q^{ir+s}}{1 - q^{ir+s}}$$

$$B^{(s)}(q) = P_r(q) \sum_{j \geq 1} \frac{q^{js} - q^{jr}}{1 - q^{jr}}$$

$$D(q) = P_r(q) \sum_{j \geq 1} \frac{q^{jr}}{1 - q^{jr}}.$$

We omit the details. From this Proposition 6 may be deduced easily.

3 Smith normal forms of character tables related to S_n

For a partition λ of n, we denote by ξ^λ the permutation character of S_n obtained by inducing the trivial character of the Young subgroup S_λ up to S_n. First we explicitly describe the values of these permutation characters (this is included here as we have not been able to find a reference for it).

Proposition 10 *Let $\lambda, \mu \in \mathcal{P}$, $k = \ell(\lambda)$, $\ell = \ell(\mu)$. Then the value $\xi^\lambda(\mu)$ of the permutation character ξ^λ on the conjugacy class of cycle type μ equals the number of ordered set partitions $(B_1, \dots, B_k)$ of $\{1, \dots, \ell\}$ such that*

$$\lambda_j = \sum_{i \in B_j} \mu_i \quad \textit{for } j \in \{1, \dots, k\} .$$

Proof. Let σ_μ be a permutation of cycle type μ. Then (see [8]) $\xi^\lambda(\mu)$ is the number of λ-tabloids fixed by σ_μ. Now clearly, a λ-tabloid is fixed by σ_μ if and only if its rows are unions of complete cycles of σ_μ. Thus such a decomposition of rows corresponds to an ordered set partition $(B_1, \dots, B_k)$ of the cycles of μ with the sum conditions in the statement of the Proposition. $\square$

Remark. One may also use a symmetric function argument for computing the values $R_{\lambda\mu} = \xi^\mu(\lambda)$. The complete homogeneous symmetric function h_λ is the (Frobenius) characteristic of the character ξ^λ (see [14, Cor. 7.18.3]), so $h_\lambda = \sum_\mu z_\mu^{-1} R_{\lambda\mu} p_\mu$. As the h_λ and m_μ are dual bases, as well as the p_λ and $z_\mu^{-1} p_\mu$, it then follows that $p_\lambda = \sum_\mu R_{\lambda\mu} m_\mu$. Using [14, Prop. 7.7.1] then also gives the formula in Proposition 10.

Corollary 11 *Let $\lambda, \mu \in \mathcal{P}$. Then we have*

(i) $\xi^\lambda(\mu) = 0$ *unless* $\lambda \geq \mu$ *(dominance order).*

(ii) $\xi^\lambda(\lambda) = b_\lambda = \prod_i m_i(\lambda)!$.

(iii) $\xi^\lambda(\lambda) \mid \xi^\lambda(\mu)$.

Proof. Using the remark above, parts (i) and (ii) follow immediately by [14, Cor. 7.7.2] (or one may also prove it directly using Proposition 10). For (iii), we use the combinatorial description given in Proposition 10. With notation as before, let $(B_1, \dots, B_k)$ be an ordered partition of the set $\{1, \dots, \ell\}$ contributing to $\xi^\lambda(\mu)$, i.e., satisfying the sum conditions. Now any permutation of $\{1, \dots, k\}$ which interchanges only parts of λ of equal size leads to a permutation of the entries of $(B_1, \dots, B_k)$ such that the corresponding ordered partition still satisfies the sum conditions. Hence $\xi^\lambda(\mu)$ is divisible by $\prod_i m_i(\lambda)! = b_\lambda$ and thus by $\xi^\lambda(\lambda)$. $\square$

We can now determine the Smith normal form for the regular character table of S_n in the case where $r = p$ is prime.

For an integer matrix A we denote by $\mathcal{S}(A)$ its Smith normal form. If p is a prime, we write $A_{p'}$ for the matrix obtained by taking only the p'-parts

of the entries. For a set of integers $M = \{r_1, \ldots, r_m\}$ we denote by $\mathcal{S}(M)$ or $\mathcal{S}(r_1, \ldots, r_m)$ the Smith normal form of the diagonal matrices with the entries $r_1, \ldots, r_m$ on the diagonal.

Theorem 12 *Let p be a prime, and let $\mathcal{X}_{RC}$ be the p-regular character table of S_n. Then we have*

$$\mathcal{S}(\mathcal{X}_{RC}) = \mathcal{S}(b_\mu \mid \mu \in C)_{p'} .$$

Proof. Let $\mathcal{Y} = \mathcal{Y}_{CC} = (\xi^\lambda(\mu))_{\lambda,\mu\in C}$ denote the part of the permutation character table of S_n with rows and columns indexed by the class p-regular partitions of n. Set $\mathcal{X} = \mathcal{X}_{RC}$.
As the characters χ^λ with λ in the set R of p-regular partitions of n form a basic set for the characters on the p-regular conjugacy classes by [9], we have a decomposition matrix $D = D_{CR}$ with integer entries such that

$$\mathcal{Y} = D \cdot \mathcal{X} .$$

Now by Corollary 11 the permutation character table $\mathcal{Y}$ is (with respect to a suitable ordering) a lower triangular matrix with the b_μ, $\mu \in C$, on the diagonal. Hence using [12, Theorem 4] and Theorem 1 we obtain

$$\det(\mathcal{Y})_{p'} = (b_C)_{p'} = a_C = |\det(\mathcal{X})| .$$

Thus $\det(D)$ is a p-power, and hence $\det(D)$ and $\det(\mathcal{X})$ are coprime. This implies by [11, Theorem II.15]

$$\mathcal{S}(\mathcal{Y}) = \mathcal{S}(D\,\mathcal{X}) = \mathcal{S}(D)\mathcal{S}(\mathcal{X}) .$$

Now using the divisibility property in Corollary 11 (iii) we can convert the triangular matrix $\mathcal{Y}$ by unimodular transformations to a diagonal matrix with the same entries b_μ, $\mu \in C$, on the diagonal, and hence $\mathcal{S}(\mathcal{Y}) = \mathcal{S}(b_\mu \mid \mu \in C)$. As $\mathcal{S}(D)$ is a diagonal matrix with only p-power entries on the diagonal, this yields the assertion in the Theorem. $\square$

Remark. Choosing $p > n$ in Theorem 12 shows in particular that the Smith normal form of the whole character table $\mathcal{X}$ is the same as that of the diagonal matrix with diagonal entries $b_\mu = R_{\mu\mu}$, $\mu \in \mathcal{P}$. One may also use the language of symmetric functions to prove this result. Here, one uses that the matrix $\mathcal{X}$ is the transition matrix from the Schur functions to the power sums [14, Cor. 7.17.4]. Since the transition matrix from the monomial symmetric functions to the Schur functions is an integer matrix of determinant 1 (in fact, lower unitriangular with respect to a suitable ordering on partitions

[14, Cor. 7.10.6]), the transition matrix $R_n = (R_{\lambda\mu})_{\lambda,\mu\in\mathcal{P}}$ between the m_λ's and p_μ's has the same Smith normal form as $\mathcal{X}$. Then we use the same arguments as before to deduce the Smith normal form of R_n.

Remark. We do not know at present how Theorem 12 should extend from the prime case to the case of general r. Some obvious guesses for r-versions do not hold. The following weaker version *might* be true. Let π be the set of primes of r, and for a number m let $m_{\pi'}$ denote its π'-part (the largest divisor of m coprime to r). Then

$$\mathcal{S}(\mathcal{X}_{RC})_{\pi'} = \mathcal{S}(b_\mu \mid \mu \in C)_{\pi'} \ .$$

Using Theorem 12 above for $p = 2$ also allows the determination of the Smith normal form of the reduced spin character table of the double covers of the symmetric groups. For the background on spin characters of S_n we refer to [7] and [13].

We denote by $\mathcal{D}$ the set of partitions of n into distinct parts and by $\emptyset$ the set of partitions of n into odd parts. Note that thus $\mathcal{D}$ is the set of 2-regular partitions of n and $\emptyset$ is the set of class 2-regular partitions of n. For each $\lambda \in \mathcal{D}$ we have a spin character $\langle\lambda\rangle$ of S_n. If $n - \ell(\lambda)$ is odd, then there is an associate spin character $\langle\lambda\rangle' = \mathrm{sgn} \cdot \langle\lambda\rangle$ of S_n and λ is said to be of negative type; the corresponding subset of $\mathcal{D}$ is denoted by $\mathcal{D}^-$. The spin characters can have non-zero values only on the so-called doubling conjugacy classes of the double cover $\tilde{S}_n$ of S_n; these are labelled by the partitions in $\emptyset \cup \mathcal{D}^-$. More precisely, for any such partition we have two conjugacy classes in $\tilde{S}_n$; one of these is chosen in accordance with [13], and we denote a corresponding representative by σ_μ. While the spin character values on the $\mathcal{D}^-$ classes are known explicitly (but they are in general not integers, and mostly not even real), for the values on the $\emptyset$-classes we only have a recursion formula (due to A. Morris) which is analogous to the Murnaghan-Nakayama formula, and which shows that these are integers. We then define the reduced spin character table as the integral square matrix

$$Z_s = (\langle\lambda\rangle(\sigma_\mu))_{\substack{\lambda\in\mathcal{D}\\ \mu\in\emptyset}}$$

For any integer $m \geq 0$, let $s(m)$ be the number of summands in the 2-adic decomposition of m. For $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \emptyset$ we define

$$k_\alpha = \sum_{i \text{ odd}} (m_i - s(m_i)) \ .$$

Then we have

Theorem 13 *The Smith normal form of the reduced spin character table Z_s of $\tilde{S}_n$ is given by*

$$\mathcal{S}(Z_s) = \mathcal{S}(2^{[k_\mu/2]}, \mu \in \emptyset) \cdot \mathcal{S}(b_\mu, \mu \in \emptyset)_{2'} \ .$$

Proof. Let Φ denote the Brauer character table of $\tilde{S}_n$ at characteristic 2; this is equal to the Brauer character table of S_n. Then $Z_s = D_s \cdot \Phi$, where D_s is a "reduced" decomposition matrix at $p = 2$; the reduction corresponds to leaving out the associate spin characters $\langle\lambda\rangle'$ for $\lambda \in \mathcal{D}^-$. The matrix D_s is then an integral square matrix. In [1], the Smith normal form of D_s was determined:

$$\mathcal{S}(D_s) = \mathcal{S}(2^{[k_\mu/2]}, \mu \in \emptyset) \ .$$

As this is a matrix of 2-power determinant and the determinant of the Brauer character table is coprime to 2, we have

$$\mathcal{S}(Z_s) = S(D_s) \cdot \mathcal{S}(\Phi) = \mathcal{S}(2^{[k_\mu/2]}, \mu \in \emptyset) \cdot \mathcal{S}(\Phi) \ .$$

Now the Brauer characters and the characters χ^λ, $\lambda \in R = \mathcal{D}$, are both basic sets for the characters of S_n on 2-regular classes, hence $\mathcal{S}(\Phi) = S(\mathcal{X}_{RC})$. By Theorem 12 (for $p = 2$) we thus obtain

$$\mathcal{S}(\Phi) = \mathcal{S}(\mathcal{X}_{RC}) = \mathcal{S}(b_\mu \mid \mu \in \emptyset)_{2'} \ .$$

This proves the claim. $\square$

Remark. Let us finally mention some open questions. We have determined the Smith normal form for the whole reduced spin character table. It is natural to ask whether also a p-version (or even an r-version) of this holds, or at least, whether the determinant can be computed similarly as in the ordinary S_n case.

More precisely, for a prime p define

$$Z_{s,p} = (\langle\lambda\rangle(\sigma_\mu))_{\substack{\lambda \in \mathcal{D}_p \\ \mu \in \emptyset_p}}$$

where $\mathcal{D}_p$ and $\emptyset_p$ denote the sets of class p-regular partitions in $\mathcal{D}$ and $\emptyset$, respectively. Some examples lead to the following conjecture:

$$\mathcal{S}(Z_{s,p}) = \mathcal{S}(2^{[k_\mu/2]}, \mu \in \emptyset_p) \cdot \mathcal{S}(b_\mu, \mu \in \emptyset_p)_{2'} \ .$$

Concerning the determinant, one may ask whether there is an analogue of Theorem 4 in the spin case.

For S_n as well as its double cover one may also try to look for sectional versions or block versions for the results on regular character tables.

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